The Likelihood ratio test statistic ~~(LRT)~~ for

testing $H_0: \theta \in \Theta_0$ v.s. $H_1: \theta \in \Theta_1$ is $\quad \Omega = \Omega_0 \cup \Omega_1$

$$\lambda(x) = \frac{\max_{\Theta_0} L(\hat{\hat{\theta}} \mid x)}{\max_{\Theta} L(\hat{\theta} \mid x)}$$

$\to$ restricted MLE under H_0

$\to$ unrestricted MLE.

A likelihood ratio test (LRT) is any test that has a R.R. of the form $\{x : \lambda(x) \le c\}$, where c is any number satisfying $0 \le c \le 1$.

$\because$ for a given data set, it's more likely ~~for~~ that θ falls into H_1

<u>Example 1</u>: Let $X_i \overset{iid}{\sim} N(\theta, 1)$. Consider testing $H_0: \theta = \theta_0$ v.s. $H_1: \theta \neq \theta_0$. Here θ_0 is a number fixed by the experimenter prior to the experiment.

UMP doesn't exist. use LRT.

$$\boxed{\begin{array}{l} \Omega_0 = \theta = \theta_0 \\ \Omega = -\infty < \theta < \infty \end{array}}$$

<u>sol</u>: $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x_i-\mu)^2/2\sigma^2} = \frac{1}{\sqrt{2\pi}} e^{-(x_i-\mu)^2/2\sigma^2}$

the restricted MLE under H_0: $\hat{\hat{\theta}} = \theta_0$

the unrestricted MLE: $\hat{\theta} = \bar{X}$

So, the LRT statistic for testing $H_0: \theta = \theta_0$ v.s. $H_1: \theta \neq \theta_0$ is.

$$\lambda(x) = \frac{(2\pi)^{-n/2} e^{-\sum_{i=1}^{n}(x_i-\theta_0)^2/2}}{(2\pi)^{-n/2} e^{-\sum_{i=1}^{n}(x_i-\bar{x})^2/2}} = \exp\left[\frac{-\sum(x_i-\theta_0)^2 + \sum(x_i-\bar{x})^2}{2}\right]$$

$$= \exp\left[\frac{-\sum x_i^2 + 2\theta_0 n\bar{x} - n\theta_0^2 + \sum x_i^2 - 2n\bar{x}^2 + n\bar{x}^2}{2}\right]$$

$$= \exp\left[\frac{2\theta_0 n\bar{x} - n\theta_0^2 - n\bar{x}^2}{2}\right] = \exp\left[\frac{-n(\bar{x}-\theta_0)^2}{2}\right] \le c$$

$\lambda(x) \le c$ $\quad \left(\because 0 < c \le 1, \therefore \frac{-2\ln c}{n} > 0\right)$

$\Rightarrow (\bar{x}-\theta_0)^2 \geq \frac{-2\ln c}{n}$

$\Rightarrow |\bar{x}-\theta_0| \geq \sqrt{\frac{-2\ln c}{n}}$ is the R.R.

Thus, the LRTs are just those tests that reject $H_0: \theta=\theta_0$ if the sample mean differs from the hypothesized value θ_0 by more than a specified amount.

For large n, $\bar{x} \sim N(\theta_0, \frac{1}{n})$ under $H_0: \theta=\theta_0$.

$\alpha = P(\text{Reject } H_0 \text{ when } H_0 \text{ is true}) = P\left(|\bar{x}-\theta_0| \geq \sqrt{\frac{-2\ln c}{n}} \text{ when } \theta=\theta_0\right)$

$= P\left(|Z| \geq \frac{\sqrt{\frac{-2\ln c}{n}}}{\sqrt{\frac{1}{n}}}\right) = P(|Z| \geq \sqrt{-2\ln c})$

$\sqrt{-2\ln c} = Z_{\frac{\alpha}{2}} \quad \Rightarrow -2\ln c = Z_{\frac{\alpha}{2}}^2$

$\Rightarrow$ LRT R.R. is $|\bar{x}-\theta_0| \geq \sqrt{\frac{Z_{\frac{\alpha}{2}}^2}{n}}$

$\Rightarrow \sqrt{n}\,|\bar{x}-\theta_0| \geq Z_{\frac{\alpha}{2}}$. (Same as large sample Z test result).

$\Rightarrow \{\text{R.R.}: \bar{x} > \theta_0 + \frac{1}{\sqrt{n}} Z_{\frac{\alpha}{2}}$ or $\bar{x} < \theta_0 - \frac{1}{\sqrt{n}} Z_{\frac{\alpha}{2}}\}$

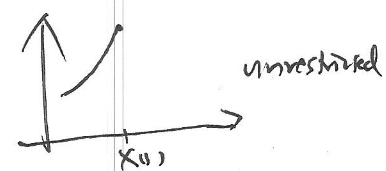
Example 2:

Let $X_1, \dots X_n \sim$ exponential population with pdf

$$f(x|\theta) = \begin{cases} e^{-(x-\theta)} & x \geq \theta \\ 0 & x < \theta. \end{cases} \quad \text{where } -\infty < \theta < \infty.$$

$\Omega_0 : \theta \leq \theta_0$
$\Omega : -\infty < \theta < \infty$

Consider LRT ~~test~~ for testing $H_0 : \theta \leq \theta_0$ v.s. $H_1 : \theta > \theta_0$

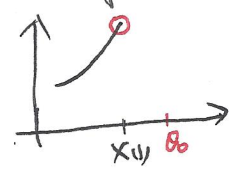


sol: $L(\theta|x) = e^{-\Sigma x_i + n\theta} I(X_{(1)} \geq \theta)$

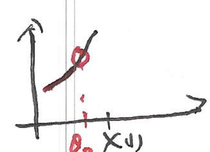
the unrestricted MLE of θ is: $\hat{\theta}_{MLE} = X_{(1)}$

the unrestricted max of $L(\theta)$ is: $L(\hat{\theta}) = e^{-\Sigma x_i + nX_{(1)}} I(X_{(1)} \geq \theta)$

Next, let's consider the restricted MLE of θ under H_0:



① if $X_{(1)} \leq \theta_0$, then $\hat{\hat{\theta}} = X_{(1)}$



② if $X_{(1)} > \theta_0$, then $\hat{\hat{\theta}} = \theta_0$

$$\lambda(x) = \begin{cases} e^{-n(X_{(1)} - \theta_0)} & \text{if } X_{(1)} > \theta_0 \\ 1 & \text{if } X_{(1)} \leq \theta_0 \end{cases}$$

A test that rejects H_0 if $\lambda(x) \leq c$ is a test

An LRT, a test that rejects H_0 with R.R. $\left\{ X_{(1)} \geq \boxed{\theta_0 - \frac{\ln c}{n}} \right\}$ k^*

Next, let's consider the value of c.

$\alpha = P(\text{Reject } H_0 \text{ when } H_0 \text{ is true}) = P\left(X_{(1)} \geq \theta_0 - \frac{\ln c}{n}\right.$ when H_0 is true$\left.\right)$

$f_{X_{(1)}}(x) = n(1-F(x))^{n-1} f(x)$

where $f(x) = e^{-(x-\theta_0)}$ and $F(x) = \int_\theta^x e^{-(t-\theta)} dt = 1 - e^{\theta - x}$

Thus, $f_{X_{(1)}}(x) = n \cdot e^{(\theta - x)(n-1)} \cdot e^{-(x-\theta)} = n e^{(\theta - x)n} = n e^{-(x-\theta)n}$

$$\alpha = P(X_{(1)} \geq K^*) = \int_{K^*}^{\infty} n e^{-nx} e^{n\theta_0} dx = n e^{n\theta_0} \frac{e^{-nx}}{(-n)} \Big|_{K}^{\infty}$$

$$= e^{n(\theta_0 - K^*)} = \alpha \quad \Rightarrow K^* = \theta_0 - \frac{\ln\alpha}{n}$$

Hence, ~~the~~ LRT R.R $= \left\{ X_{(1)} \geqslant \theta_0 - \frac{\ln\alpha}{n} \right\}$

Example 3:

Suppose that $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$ with unknown mean μ and unknown variance σ^2. Use LRT to consider

$H_0: \mu = \mu_0$ v.s. $H_1: \mu > \mu_0$

$$\boxed{\begin{array}{l}\Omega_0: \mu = \mu_0 \\ \Omega: \mu \geq \mu_0\end{array}}$$

sol: ~~From previous practice, we know that the unrestricted MLE of σ^2 is: $\hat{\sigma}^2 = \frac{\sum (x_i - \bar{x})^2}{n}$~~

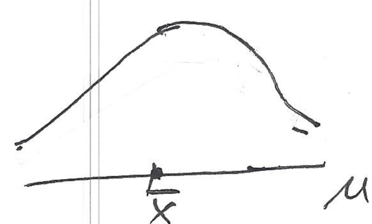
$$L(\theta) = L(\mu, \sigma^2) = (2\pi)^{-\frac{n}{2}} (\sigma^2)^{-\frac{n}{2}} \exp\left(-\sum_{i=1}^{n} \frac{(x_i - \mu)^2}{2\sigma^2}\right)$$

$$\ln L(\theta) = -\frac{n}{2} \ln \sigma^2 - \frac{n}{2} \ln 2\pi - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2$$

$$\frac{\partial \ln L(\theta)}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu)$$

$$\left\{ \frac{\partial \ln L(\theta)}{\partial \sigma^2} = -\left(\frac{n}{2\sigma^2}\right) + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (x_i - \mu)^2 \right.$$

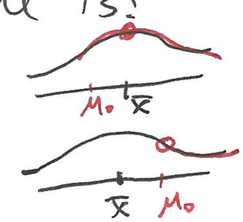
Note that
$$\begin{cases} \partial L(\theta)/\partial \mu < 0 & \text{if } \mu > \bar{x} \\ \partial L(\theta)/\partial \mu = 0 & \text{if } \mu = \bar{x} \\ \partial L(\theta)/\partial \mu > 0 & \text{if } \mu < \bar{x} \end{cases}$$



Hence, the unrestricted MLE of μ is: (H_0 and H_1 combined.)

($\Omega: \{(\mu, \sigma^2): \mu \geq \mu_0, \sigma^2 > 0\}$)

$$\hat{\mu} = \begin{cases} \bar{X} & \text{if } \bar{X} > \mu_0 \\ \mu_0 & \text{if } \bar{X} \leq \mu_0 \end{cases}$$



the unrestricted MLE of σ^2 is

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} (x_i - \hat{\mu})^2}{n}$$

$$L(\Omega) = L(\hat{\Theta}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\hat{\sigma}^2}\right)^{n/2} \exp\left[-\sum \frac{(x_i - \hat{\mu})^2}{2\hat{\sigma}^2}\right] = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\hat{\sigma}^2}\right)^{n/2} e^{-n/2}.$$

Meanwhile, the restricted MLE of μ is:

$$\hat{\hat{\mu}} = \mu_0 \quad \text{and} \quad \hat{\hat{\sigma}}^2 = \frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{n}$$

or $L(\Omega_0)$

$$L(\hat{\hat{\Theta}}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\hat{\hat{\sigma}}^2}\right)^{n/2} \cdot \exp\left[-\sum \frac{(x_i-\mu_0)^2}{2\hat{\hat{\sigma}}^2}\right] = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\hat{\hat{\sigma}}^2}\right)^{n/2} e^{-n/2}$$

Thus, $$\lambda(x) = \frac{L(\hat{\hat{\Theta}})}{L(\hat{\Theta})} \overset{\text{or}}{=} \frac{L(\Omega_0)}{L(\Omega)} = \left(\frac{\hat{\sigma}^2}{\hat{\hat{\sigma}}^2}\right)^{n/2}$$

$$= \begin{cases} \left[\dfrac{\sum(x_i-\bar{x})^2}{\sum(x_i-\mu_0)^2}\right]^{n/2} & \text{if } \bar{x} > \mu_0 \\ 1 & \text{if } \bar{x} \le \mu_0 \end{cases}$$

$$\lambda(x) < c \Rightarrow \lambda(x) = \left[\frac{\sum(x_i-\bar{x})^2}{\sum(x_i-\bar{x})^2 + n(\bar{x}-\mu_0)^2}\right]^{n/2} < c$$

$$\Rightarrow \frac{1}{1+\dfrac{n(\bar{x}-\mu_0)^2}{\sum(x_i-\bar{x})^2}} < c^* \Rightarrow \frac{n(\bar{x}-\mu_0)^2}{\sum(x_i-\bar{x})^2} > c^{**}$$

$$\Rightarrow \frac{n(\bar{x}-\mu_0)^2}{\dfrac{\sum(x_i-\bar{x})^2}{n-1}} > (n-1)c^{**} \Rightarrow \frac{n(\bar{x}-\mu_0)^2}{s^2} > c^{***}$$

$$\Rightarrow \boxed{\frac{\bar{x}-\mu_0}{s/\sqrt{n}} > \sqrt{c^{***}}}$$

this is the t-test we used in 10.8.